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Quasi-Stationary States for Particle Systems in the Mean-Field Limit

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Abstract We prove that the N particles approximation of a class of stable stationary solutions of the Vlasov equation is uniformly valid on a time scale N^{β} for $\beta > 0$ (explicitly given in various cases) much longer than the usual $\log N$ scale. The vortex blob method in dimension 2 is also discussed. The result applies to a class of stationary solutions more general than in a previous work.

Keywords N-particle systems · Vlasov equation · Long-time estimates · Metastability

1 Introduction

We consider the Vlasov type equation

$$\partial_t f + v \cdot \nabla_x f + F[\rho_f] \cdot \nabla_v f = 0, \tag{1}$$

where $f(t, x, v) \geq 0$, $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, the density ρ_f and the force $F[\rho_f]$ are given by

$$\rho_f(t, x) = \int_v f(t, x, v) dv,$$

$$F[\rho_f](t, x) = -\nabla V \star \rho_f = -\int_y \nabla V(x - y) \rho_f(t, y) dy,$$

and we assume that the potential V is periodic and such that $V(x) = V(-x)$, $\int_{\mathbb{T}^d} V = 0$. This equation describes the dynamics of N particle system in the mean-field limit. This

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assertion can be formalized in the following way: let us consider N particles with positions x_i, $i = 1, \dots, N$ and velocities v_i, $i = 1, \dots, N$, which weakly interact via the two-body potential V:

$$\dot{x}_i = v_i, \tag{2}$$

$$\dot{v}_i = -\frac{1}{N}\sum_{j=1}^{N} \nabla V(x_i - x_j). \tag{3}$$

We can associate to this particle system the empirical measure

$$\mu_N(t) = \frac{1}{N}\sum_{k=1}^{N} \delta_{(x_k(t), v_k(t))} \tag{4}$$

and we expect that if $\mu_N(0)$ converges weakly to f_0 then the empirical measure $\mu_N(t)$ converges, also weakly, to the solution of (1) with f_0 as initial data. When V is a singular potential like a Newtonian or Coulombian potential for $d = 2$, or $d = 3$ the rigorous justification of the previous assertion remains a widely open problem despite a recent advance for some weakly singular potential (i.e $|\nabla V| \le C|x|^{-\alpha}$ for $\alpha < 1$) [16]. We shall not work in this direction here, we shall focus on some qualitative questions which remain open even when V is smooth. When $F = -\nabla V$ is Lipschitz, the justification is easier. Indeed there is no problem to define measure solutions of the Vlasov equation (1), we refer to [25] for example. Moreover the empirical measure $\mu_N(t)$ is actually a measure solution of the Vlasov equation (1) so that the justification of the large N limit can be obtained from a Lipschitz estimate for the flow of (1) defined on the probability measures. To state a precise result we first need to define some distance on the space of probability measures. We set

$$\|\mu\|_{BL^*} = \sup_{\|\varphi\|_{BL} \le 1} \int_{x,v} \varphi(x, v)\, d\mu(x, v),$$

where

$$\|\varphi\|_{BL} = \|\varphi\|_{L^\infty} + \|\varphi\|_{\mathrm{Lip}}$$

with

$$\|\varphi\|_{\mathrm{Lip}} = \sup_{(x_1,v_1) \ne (x_2,v_2)} \frac{|\varphi(x_1, v_1) - \varphi(x_2, v_2)|}{|x_1 - x_2| + |v_1 - v_2|}.$$

The relevance of this norm is that the distance induced by the norm $\|\cdot\|_{BL^*}$ on the probability measures metricizes the weak convergence. This distance is actually equivalent to a Wasserstein distance but we shall not use this here.

With respect to this distance, the following classical result holds (see, for instance [6, 13, 22, 25]):

Theorem 1 *Assume that ∇V is Lipschitz, then there exists $C > 0$, $\kappa > 0$ such that for any bounded measures μ_1^0, μ_2^0, the two measure solutions of (1) $\mu_1(t)$, $\mu_2(t)$ with initial data μ_1^0 and μ_2^0 satisfy the estimate*

$$\|\mu_1(t) - \mu_2(t)\|_{BL^*} \le C e^{\kappa t} \|\mu_1^0 - \mu_2^0\|_{BL^*}, \quad \forall t \ge 0. \tag{5}$$

Consequently, if $\mu_N(0)$ converges to μ_0 in the bounded Lipschitz norm of measures (equivalent to the weak convergence), then for every fixed time t, $\mu_N(t)$ also converges to $\mu(t)$, the solution of (1), with initial value μ_0. Moreover the convergence is uniform on every finite interval of time $[0,T]$. To be more precise, let us assume that the initial positions and velocities $(x_i^0, v_i^0)_{i\in\mathbb{N}}$ of the particles are independent identically distributed random variables with law $\mu(0)$. Then in the simplest case where $d=1$ (we shall come back to the general case later), for every $\epsilon>0$, there exists $C>0$ such that

$$\|\mu_N(0)-\mu(0)\|_{BL^*}\le \frac{C}{N^{1/2-\epsilon}} \tag{6}$$

with probability one (a precise statement of the result is given below in Lemma 2). Consequently, by applying Theorem 1 we obtain

$$\|\mu_N(t)-\mu(t)\|_{BL^*}\le e^{\kappa t}\frac{C}{N^{1/2-\epsilon}}$$

and therefore $\|\mu_N(t)-\mu(t)\|$ remains uniformly of order $\frac{1}{N^\alpha}$ for every $\alpha\in(0,1/2)$ on a time scale of order $\log N$. Moreover, for finite times, the fluctuations of the system (2, 3) have been fully characterized and shown to evolve according to the Vlasov equation linearized around $\mu(t)$. In particular propagation of molecular chaos holds in the mean-field limit and the fluctuations of the observables are described by a Gaussian process (see [6, 25]). Consequently, for finite times, the behaviour of the particles systems is very well understood. Then, the natural question that arises is:

what happens for longer times?

The problem of describing the $o(N)$ corrections to Vlasov type equations (1) in order to describe the dynamics of weakly interacting N particles systems (2, 3) on a longer time scale has been considered in the physical literature for a long time and remains an active field of research. For example, in the recent papers [4, 5, 27], a simple model of weakly interacting particles on a circle which is called the Hamiltonian Mean-Field Model (HMF) has been studied. This model corresponds to the potential $V=\cos(x)$ in our general framework. Note that this potential is smooth and hence in the framework of Theorem 1. The formal idea is that the long time behaviour should be described by an equation like

$$\partial_t f + v\cdot\nabla_x f + F[\rho_f]\cdot\nabla_v f = \epsilon(N)Q(f), \tag{7}$$

where $\epsilon(N)$ is a coefficient vanishing with N in a suitable way. A possible scenario, for the dynamics is that in a first stage, f will be driven towards the manifold of the stable stationary solutions of the Vlasov equation (1) and then in a second stage, f will slowly evolve in the vicinity of this manifold until it reaches eventually the Maxwellian equilibrium. In [27] the scenario described above is proposed and analyzed from a physical point of view. In particular the authors give some theoretical arguments supporting the fact that the time to approach the thermodynamic equilibrium is larger than $O(N)$. Furthermore, from numerical simulations, they get that the system should converge to the thermodynamic equilibrium in times $T_N\simeq N^\alpha$, with a non trivial exponent $\alpha\simeq 1.7$. A similar scenario is exhibited by the so called "adiabatic piston" model, see [7, 10–12, 15, 18]. Also in this case relaxation to equilibrium is reached in times diverging with a power of N where N is the number of particles.

In the above formal considerations, the understanding of the dynamics in the vicinity of the manifold of the stable stationary solutions of (1) seems to play an important role towards a rigorous understanding. Motivated by this fact, we shall focus on this part of the proposed scenario: we analyze rigorously the dynamics of the particles system in the vicinity of a stable stationary solution of Vlasov. We do not know how to describe a correction term to Vlasov but if the scenario proposed is valid, it seems reasonable that if at the initial time the particles system is close to a stationary stable solution of the Vlasov equation (1) f_0, then the solution of the particles system will remain close to f_0 for times much longer than $O(\log N)$. In other words the stable stationary solutions of the Vlasov equations can be seen as metastable states for the particle system. This is indeed what we prove in a mathematically rigorous way: we show that if initially the particles are close to f_0 in the sense of (6), then the particles remains close to f_0 uniformly at least on a time scale of order N^α, where $\alpha > 0$ is explicitly given. For example if f_0 is the electron patch distribution, $f_0 = \mathbf{1}_{|v|\leq 1}$, we can get $\alpha = 1/10$.

Let us be more specific about our result and its proof. An interesting class of stationary solutions of (1) is given by $f_0 = g(|v|)$. We know, thanks to the Penrose criterion [23], that if g is a non increasing function of $|v|$ then it is a stable stationary solution of the Vlasov equation. Assuming in addition that f_0 is sufficiently smooth (C^2), the stability of a system of particles randomly extracted from f_0 was already studied in a previous paper [8]. It was shown that the particles system remains close to the stationary distribution f_0 on a time scale $N^{\frac{1}{8}}$. The main ingredient in our proof was the Energy-Casimir method which was introduced by Arnold for the Euler equations [1] and then generalized to various equations with Hamiltonian structure by Holm, Marsden, Ratiu and Weinstein [17], see also [2, 24]. The main drawback of this method is that it requires some smoothness of $f_0(C^2)$ and in particular the result of [8] does not cover the case where f_0 is an electron patch distribution $f_0 = \mathbf{1}_{|v|\leq 1}$. Here we study the more general case where f_0 is any non increasing compactly supported L^∞ function of $|v|$, without any regularity requirement. It was shown by Marchioro and Pulvirenti [21] (see also [3]), that these solutions are indeed stable solutions of the Vlasov equation (1) (actually their result is also valid for the Vlasov–Poisson equation) in the sense that small perturbations of f_0 in $L^1_{x,v}$ remain close to f_0 in $L^1_{x,v}$ for ever. The aim of this paper is in some sense to get long time estimates for perturbations of f_0 in a much more general class of perturbations of f_0 which is the class of bounded measures. By applying the result to the empirical measures (4), we are able to prove that the particle system remains close to f_0 up to times of the order of $N^{1/14}$. Moreover for a particular class of stationary solutions f_0, which in particular includes the electron patch $\mathbf{1}_{|v|\leq 1}$, we can reach times of the order of $N^{1/10}$.

These estimates are just bounds from below and are very probably not optimal. Nevertheless, it is also easy to get an upper bound of the best possible time scale. Indeed, we emphasize that our results are deterministic in the sense that we do not exclude some possibly measure zero sets of initial data which are bad for the dynamics: we prove our result for every initial sequence (x_i^0, v_i^0) of initial positions and velocities, which approximates f_0 in the bounded Lipschitz norm of measure with error $N^{-\frac{1}{2}+\epsilon}$. With no further hypothesis on the initial data we know that we cannot prove our result for too long times. In particular in the case of the free transport (i.e when $V = 0$), we can construct an explicit example for which the particles system is at distance $O(1)$ from the uniform distribution after a time $O(N^{\frac{1}{2}})$. The construction is given in [8]. Consequently without any additional restriction on the choice of the initial date we cannot get in general estimates on a time scale longer that $N^{\frac{1}{2}}$.

The paper is organized as follows: in Sect. 2 we state and prove our result when $d = 1$, in Sect. 3, we give the extensions of the result in higher dimensions. In Sect. 4, we discuss the problem of long time estimates in the vortex blob method for the incompressible Euler equation which is a strongly related problem.

2 The One-Dimensional Case

Let $f_0(v) \geq 0$, essentially bounded and compactly supported such that

$$\int_{x,v} f_0 dx dv = 1,$$

we can consider the probability measure

$$P = \otimes_{\mathbb{N}} f_0 dx dv$$

on $(\mathbb{T} \times \mathbb{R})^{\mathbb{N}}$ and consider the initial sequences $(x_i^0, v_i^0)_{i\in\mathbb{N}}$ as random variables. We have the following Lemma:

Lemma 2 *For every $\alpha \in (0, 1/2)$ there exists a set $E \subset (\mathbb{T} \times \mathbb{R})^{\mathbb{N}}$ with $P((\mathbb{T} \times \mathbb{R})^{\mathbb{N}} \backslash E) = 0$ and a constant $C > 0$ such that for every sequence $(X_i(0) = (x_i(0), v_i(0))_{i\in\mathbb{N}}) \in E$, we have for every bounded Lipschitz function ψ:*

$$\left| \frac{1}{N} \sum_{i=1}^{N} \psi(X_i(0)) - \int_{x,v} \psi(x, v) f_0(v) dx dv \right| \leq \frac{C \|\psi\|_{BL^*}}{N^{\frac{1}{2}-\alpha}}. \tag{8}$$

In other words this lemma says that almost every choice of the initial particles is a good approximation of f_0 in the bounded Lipschitz distance of measures, since (8) is equivalent to

$$\|\mu_N(0) - f_0 dx dv\|_{BL^*} \leq \frac{C}{N^{\frac{1}{2}-\alpha}}, \tag{9}$$

where

$$\mu_N(0) = \sum_{i=1}^{N} \delta_{(x_i^0, v_i^0)}.$$

For an elementary proof of this Lemma see [8]. It is probably possible to deduce this result from some general theorem of statistics, we refer to [26]. We consider the case where f_0 is compactly supported so that, without loss of generality (this means that we can remove to E a set of measure zero), we can assume that

$$E \subset \mathbb{T}^{\mathbb{N}} \times \mathrm{Supp}(f_0)^{\mathbb{N}} \subset \mathbb{T}^{\mathbb{N}} \times [-S, S]^{\mathbb{N}} \tag{10}$$

for some $S > 0$.

The main question that we want to investigate is what happens to a particle system such that the initial positions–velocities $(x_i^0, v_i^0)_{i\in\mathbb{N}} \in E$. We can prove

Theorem 3 *Let us assume that* $f_0 = g(|v|) \in L^\infty$ *is compactly supported with* g *nonincreasing, and that* $\partial_x V \in \mathcal{C}^{1,1}(\mathbb{T})$ *is such that* $\hat{V}_l \geq 0$ *where* $\hat{V}_l$ *are the Fourier coefficients of* V.

Then there exists $C_1 > 0$, $C_2 > 0$ *and* N_0 *such that for every* $N \geq N_0$ *and* $(x_i^0, v_i^0)_{i\in\mathbb{N}} \in E$, *we have the estimate*

$$\|\mu_N(t) - f_0 dx dv\|_{BL^*} \leq C_1 \left(\frac{M_0^{\frac{1}{3}}}{N^{\frac{1}{6}-\frac{\alpha}{3}}} + \frac{M_0^{\frac{1}{3}} t^{\frac{1}{3}}}{N^{\frac{1}{6}-\frac{\alpha}{3}}} + \frac{t}{N^{\frac{1}{2}-\alpha}} \right), \quad \forall t \in [0, T_N^*] \tag{11}$$

with

$$T_N^* \geq C_2 \frac{N^{\frac{1}{14}-\frac{\alpha}{7}}}{M_2^{\frac{3}{7}} M_0^{\frac{1}{7}}},$$

where $M_0 = \|\partial_x V\|_{L^\infty}$, $M_2 = \|\partial_x V\|_{\mathcal{C}^{1,1}}$.

Roughly speaking (11) says that $\|\mu_N(t) - f_0 dx dv\|_{BL^*}$ remains of order $N^{-1/7}$ at least on a time scale of order $N^{1/14}$. In Theorem 3 the assumption on the sign of the Fourier coefficients of V means that we consider repulsive particles. This result can be seen as a first step towards the rigorous understanding of the papers [4, 5, 27] since the regularity assumptions that are needed in Theorem 3 are obviously met when $V(x) = \cos x$. Moreover, even if it was not the main motivation of this paper, we have tracked the dependence of our estimates in the regularity of V so that our Theorem can be applied to regularizations of the Vlasov–Poisson equation as it is often used in numerical methods. For Vlasov–Poisson, the potential is given by

$$\hat{V}_k = \frac{1}{|k|^2},$$

so that we can take for example as a basic regularization a finite number of Fourier modes

$$V = \sum_{1 \leq |k| \leq \varepsilon^{-1}} \frac{e^{ik\cdot x}}{|k|^2},$$

so that in this case

$$M_0 = C, \qquad M_2 = \frac{C}{\varepsilon^2},$$

and

$$T_N^* \geq C \epsilon^{\frac{6}{7}} N^{\frac{1}{14}-\frac{\alpha}{7}}.$$

2.1 Proof of Theorem 3

We shall denote by $X = (x, v)$ a point in the phase space and by

$$\mathcal{F}(t, x) = -\frac{1}{N} \sum_{i=1}^{N} \partial_x V(x - x_i(t)) \tag{12}$$

the force field created by the particles. For a given system of particles $X^N = (x_i, v_i)_{i=1,\dots,N}$, we define its kinetic and potential energy as

$$E_K(X^N(t)) = \frac{1}{N}\sum_{i=1}^{N}\frac{v_i^2}{2},$$

$$E_p(X^N(t)) = \frac{1}{N^2}\sum_{1\le i<j\le N} V(x_i(t)-x_j(t)).$$

In a similar way, for a density distribution $f(t,x,v)$, we define its kinetic energy as

$$E_K(f(t)) = \frac{1}{2}\int_{x,v} f(t,x,v)v^2 dv.$$

The main difficulty in the proof is to get some uniform bounds on the force $\mathcal{F}$ which are uniformly valid on large interval of times. Indeed, we shall see that all other important quantities are well-controlled if $\mathcal{F}$ is controlled.

At first, we notice that, since V is periodic, we have

$$\int_{x,v} \partial_x^k V(x) f_0(v) dx dv = 0, \quad k = 1, 2. \tag{13}$$

Consequently, we can rewrite $\partial_x^k \mathcal{F}(t,x)$ for $k = 0, 1$ as

$$\partial_x^k \mathcal{F}(t,x) = -\langle \mu_N(t) - f_0 dx dv, \partial_x^{k+1} V(x-\cdot)\rangle \tag{14}$$

where $\langle\cdot,\cdot\rangle$ is the duality bracket between bounded measures and continuous bounded functions.

Since at $t = 0$, we have by assumption $(x_i^0, v_i^0)_{i\in\mathbb{N}} \in E$, we can use (9) and (14) to get the initial estimate

$$\|\mathcal{F}(0,\cdot)\|_\infty + \|\partial_x \mathcal{F}(0,\cdot)\|_\infty \le \frac{CM_2}{N^{\frac{1}{2}-\alpha}}. \tag{15}$$

The proof is based on a bootstrap argument on

$$\eta(t) = \|\mathcal{F}(t,\cdot)\|_\infty + \|\partial_x \mathcal{F}(t,\cdot)\|_\infty. \tag{16}$$

In particular, at first, we will prove the following lemma, which will immediately imply Theorem 3.

Lemma 4 *Let be* $T > 0$ *and* $\epsilon > 0$ *such that*

$$\epsilon T^2 + \epsilon T < 1, \tag{17}$$

and

$$\epsilon > \frac{CM_2}{N^{\frac{1}{2}-\alpha}}. \tag{18}$$

If for any $t \in [0,T]$ *it holds*

$$\eta(t) \le \epsilon, \tag{19}$$

then for any $t \in [0,T]$ *it holds*

$$\eta(t) \le CM_2\left(\frac{1+t}{N^{\frac{1}{2}-\alpha}} + M_0^{\frac{1}{3}}\frac{1+t^{\frac{1}{3}}}{N^{\frac{1}{6}-\frac{\alpha}{3}}}\right). \tag{20}$$

2.1.1 *Proof of Lemma 4*

In the following, $C>0$ stands for a number which may change from lines to lines but which is independent of the crucial parameters in the problem: that is to say N, T, ε, and the constants describing the regularity of V, M_0, M_1, M_2.

In order to estimate η we will estimate $\|\mu_N(t) - f_0 dxdv\|_{BL^*}$, and then deduce an estimate of $\mathcal{F}$ (and therefore of η) by taking $\partial_x V$ and $\partial_{xx} V$ as test functions. In order to do this, we consider $f(t,x,v)$ the solution of the Liouville equation

$$\partial_t f + v\partial_x f + \mathcal{F}(t,x)\partial_v f = 0 \tag{21}$$

with the initial condition $f(0,x,v) = f_0(v)$. Note that f is a L^∞ function and not a measure. For every $t \in [0,T]$ we have

$$\begin{aligned}
&\|\mu_N(t) - f_0 dxdv\|_{BL^*}\\
&\quad \le \|\mu_N(t) - f(t,x,v)dxdv\|_{BL^*} + \|f(t,x,v)dxdv - f_0(v)dxdv\|_{BL^*}\\
&\quad \le I + II
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
I &= \|\mu_N(t) - f(t,x,v)dxdv\|_{BL^*},\\
II &= \|f(t,x,v)dxdv - f_0(v)dxdv\|_{BL^*}.
\end{aligned}$$

We begin with the estimate of I: for every test function $\varphi \in BL$, we have to estimate

$$A(\varphi) = \frac{1}{N}\sum_{i=1}^{N}\varphi(X_i(t)) - \int_{x,v}\varphi(x,v)f(t,x,v)dxdv. \tag{23}$$

Let us define $\Phi_t : \mathbb{T}\times \mathrm{Supp}(f_0) \to \mathbb{T}\times\mathbb{R}$, as $\Phi_t(X_0) = (x(t,X_0), v(t,X_0))$, with (x,v) the solution of the ordinary differential equation

$$\dot{x} = v, \qquad \dot{v} = \mathcal{F}(t,x) \tag{24}$$

with initial condition $X(0,X_0) = (x_0,v_0)$.

In particular, let us notice that

$$X_i(t) = \Phi_t(X_i(0)). \tag{25}$$

We point out that since f_0 is compactly supported we can restrict the flow to the support of f_0. It will be convenient to introduce $S>0$ such that $\mathrm{Supp}(f_0) \subset [-S,S]$.

Notice that f in (21) is constant along the integral curves of the flow defined by (24). Therefore the solution of (21) can be written as

$$f(t,X) = f_0(\Phi_t^{-1}(X)). \tag{26}$$

Finally, the last useful property of the flow of (24) is that it preserves the Lebesgue measure. Indeed thanks to (12) and (24), Φ_t is actually the flow of the (nonautonomous) Hamiltonian system

$$\partial_t \Phi_t = \nabla^{\perp} H(t, \Phi_t),$$

where

$$H(t, X) = \frac{v^2}{2} + \mathcal{V}(t, x),$$

$$\mathcal{V}(t, x) = \frac{1}{N}\sum_{k=1}^{N} V(x - x_k(t)).$$

Consequently, thanks to (25), (26), we can rewrite $A(\varphi)$ as

$$\begin{aligned} A(\varphi) &= \frac{1}{N}\sum_{i=1}^{N} \varphi \circ \Phi_t(X_i(0)) - \int_{x,v} \varphi(x, v) f_0(\Phi_t^{-1}(x, v))dxdv \\ &= \frac{1}{N}\sum_{i=1}^{N} \varphi \circ \Phi_t(X_i(0)) - \int_{x,v} \varphi \circ \Phi_t(x, v) f_0(v)dxdv. \end{aligned} \tag{27}$$

To get the last line, we have used that Φ_t preserves the Lebesgue measure. Looking at (27), it is very tempting to use the fact that $(X_i^0)_{i\in\mathbb{N}} \in E$ and (8) with the test function $\psi = \varphi \circ \Phi_t$. Towards this, an estimate of $\|\Phi_t\|_{BL}$ is needed. This is the aim of the following Lemma:

Lemma 5 *Assuming* (19), *we have that*

$$|v(t, X_0)| \leq S + 1, \qquad \|\Phi_t\|_{BL} \leq C(1 + t), \quad \forall t \in [0, T]. \tag{28}$$

The proof of Lemma 5 is postponed to the Appendix.

Now, in order to estimate $A(\phi)$ we use (8) with the test function $\psi = \varphi \circ \Phi_t$. Indeed, thanks to (28) we have

$$\|\varphi \circ \Phi_t\|_{BL} \leq \|\varphi\|_{BL}\|\Phi_t\|_{BL} \leq C(1 + t)\|\varphi\|_{BL}$$

for every $t \in [0, T]$ and hence, by using (8) we get

$$|A(\varphi)| \leq \frac{C\|\varphi\|_{BL}(1 + t)}{N^{\frac{1}{2}-\alpha}}, \quad \forall t \in [0, T]. \tag{29}$$

Finally (29) and the definition of the BL^* norm for measures give for I in (22) the estimate

$$I \leq \frac{C(1 + t)}{N^{\frac{1}{2}-\alpha}}, \quad \forall t \in [0, T]. \tag{30}$$

We now turn to the estimate of II. We first notice that since f is a function we have

$$II = \|f(t, x, v)dxdv - f_0(v)dxdv\|_{BL^*} \leq \|f(t) - f_0\|_{L^1_{x,v}}$$

so that it suffices to estimate this last quantity. We use again the fact that Φ_t is Lebesgue measure preserving. Indeed, let us denote by $\mathrm{Leb}(A)$ the Lebesgue measure of a measurable set A, we have

$$\mathrm{Leb}\{(x,v),\, f(t,x,v)\geq\gamma\}=\mathrm{Leb}\{(x,v),\, f_0(v)\geq\gamma\} \tag{31}$$

for every $\gamma\geq 0$, since f is given by (26). This allows us to use the following lemma due to Marchioro and Pulvirenti [20]:

Lemma 6 *If $f_0(v)$ verifies the assumptions of Theorem 3 there exists $C>0$ such that for every $h(x,v)$ with the property that*

$$\mathrm{Leb}\{(x,v), h(x,v)\geq\gamma\}=\mathrm{Leb}\{(x,v),\, f_0(v)\geq\gamma\}$$

for every $\gamma\geq 0$ then we have

$$\|h-f_0\|^3_{L^1(x,v)}\leq C(E_K(h)-E_K(f_0)). \tag{32}$$

This lemma is just a restatement of Lemma 3 in [20]. For the sake of completeness we sketch the proof of a special case of this lemma in the Appendix. This lemma means that among the functions which verifies the property (31), the minimum of the kinetic energy is reached at f_0. Of course the fact that $f_0=g(|v|)$ with g non increasing is a crucial assumption.

Thanks to (31), we can apply Lemma 6 to $f(t,x,v)$, we get

$$\|f(t)-f_0\|^3_{L^1(x,v)}\leq C(E_K(f)(t)-E_K(f_0)),\quad \forall t\geq 0 \tag{33}$$

and hence

$$II^3\leq\|f(t)-f_0\|^3_{L^1(x,v)}\leq (E_K(f)(t)-E_K(f_0)),\quad \forall t\geq 0. \tag{34}$$

We still need to estimate the right hand side of (34). We write

$$\begin{aligned}&E_K(f(t))-E_K(f_0)\\ &\quad\leq E_K(f(t))-E_K(X^N(t))+E_K(X^N(t))-E_K(f_0)=D_1+D_2\end{aligned} \tag{35}$$

with

$$\begin{aligned}D_1&=E_K(f(t))-E_K(X^N(t)),\\ D_2&=E_K(X^N(t))-E_K(f_0).\end{aligned}$$

Note that D_1 and D_2 are not necessarily positive but that their sum is because of (33). The estimate of D_1 follows easily. Indeed, we have by definition

$$D_1\leq|D_1|=\left|\frac{1}{N}\sum_{i=1}^N\frac{v_i(t)^2}{2}-\int_{x,v}f(t,x,v)\frac{v^2}{2}dxdv\right|.$$

Thanks to (28) in Lemma 5, we notice that $X_i(t)$ given by (25) remains uniformly on $[0,T]$ in a fixed compact, and also that f defined by (26) has a support in a fixed compact in velocity when T verifies (17). Consequently, we can replace in the definition of D_1 the function v^2 by the function $\varphi(x,v)=v^2\chi(v)$ where $\chi(v)$ is a smooth compactly supported function such that $\chi=1$ for $|v|\le S+1$ so that $|D_1|$ becomes

$$\left|\left\langle \mu_N(t)-f_0dxdv,\frac{v^2}{2}\chi(v)\right\rangle\right|.$$

Since $\varphi(x,v)=v^2\chi(v)/2$ is a bounded Lipschitz function we get that

$$\begin{aligned}D_1&\le|D_1|\\&\le\|\mu_N(t)-f(t,x,v)dxdv\|_{BL^*}\|v^2\chi(v)\|_{BL}\\&\le\frac{C(1+t)}{N^{\frac12-\alpha}},\quad\forall t\in[0,T]\end{aligned}\tag{36}$$

thanks to the estimate (30). It remains to estimate D_2. We notice that the potential energy $E_p(X^N(t))$ of the particle system is nonnegative. Indeed, thanks to the Fourier series expansion of the potential $V(x)$, we have

$$\begin{aligned}E_p(X^N(t))&=\frac{1}{N^2}\sum_{i,j}V(x_i-x_j)\\&=\frac{1}{N^2}\sum_l V_l\sum_{k,j}e^{il(x_k-x_j)}\\&=\frac{1}{N^2}\sum_l V_l\Big|\sum_j e^{ilx_j}\Big|^2\ge 0,\end{aligned}$$

since in the Fourier expansion

$$V(x)=\sum_{l\in\mathbb{Z}}V_le^{ilx},$$

the Fourier coefficients V_l are nonnegative by assumption. This yields

$$\begin{aligned}E_K(X^N(t))&\le E_K(X^N(t))+E_p(X^N(t))\\&=E_K(X^N(0))+E_p(X^N(0)).\end{aligned}$$

Indeed, for the system (2, 3) the total energy $E(X^N)=E_K(X^N)+E_p(X^N)$ does not depend on time. Consequently, we have

$$D_2\le E_K(X^N(0))-E_K(f_0)+E_p(X^N(0)).\tag{37}$$

To estimate D_2, we can again use that

$$E_K(X^N(0))-E_K(f_0)=\langle\mu_N(0)-f_0(v)dxdv,v^2\chi(v)\rangle,$$

and hence since $(X_i(0))_i\in E$, (9) gives

$$|E_K(X^N(0))-E_K(f_0)|\le\frac{C}{N^{\frac12-\alpha}}.\tag{38}$$

Moreover, we can write that

$$E_p(X^N(0)) = \langle \mu_N(0), \mathcal{V}(0,\cdot)\rangle,$$

with

$$\mathcal{V}(0,x) = \frac{1}{N}\sum_{k=1}^{N} V(x - x_k(0)),$$

and we notice that

$$\begin{aligned}\int_{x,v} \mathcal{V}(0,x) f_0(v)dxdv &= \left(\int_v f_0(v)dv\right)\left(\frac{1}{N}\sum_{k=1}^{N}\int_x V(x-x_k(0))dx\right)\\ &= \int_v f_0(v)dv \int_x V(x)dx\\ &= 0,\end{aligned}$$

since V is by assumption periodic with zero mean. Consequently, we can write $E_p(X^N(0))$ as

$$E_p(X^N(0)) = \langle \mu_N(0) - f_0dxdv, \mathcal{V}(0,\cdot)\rangle$$

and hence since $(X_i^0)_i \in E$ we get again from (9) that

$$|E_p(X^N(0))| \le \frac{CM_0}{N^{\frac{1}{2}-\alpha}}. \tag{39}$$

The combination of (37–39) gives the estimate for D_2:

$$D_2 \le \frac{CM_0}{N^{\frac{1}{2}-\alpha}}. \tag{40}$$

By collecting the inequalities (34–36, 40), we get that

$$II \le C\left(\frac{1+t}{N^{\frac{1}{2}-\alpha}} + M_0^{\frac{1}{3}}\frac{1+t^{\frac{1}{3}}}{N^{\frac{1}{6}-\frac{\alpha}{3}}}\right), \quad \forall t \in [0,T]. \tag{41}$$

Finally, thanks to (22) and (30), (41), we find

$$\|\mu_N(t) - f_0dxdv\|_{\text{Lip}} \le C\left(\frac{1+t}{N^{\frac{1}{2}-\alpha}} + M_0^{\frac{1}{3}}\frac{1+t^{\frac{1}{3}}}{N^{\frac{1}{6}-\frac{\alpha}{3}}}\right), \quad \forall t \in [0,T]. \tag{42}$$

By using (14) and (42), we finally get (20).

2.2 End of the Proof of Theorem 3

The proof of Theorem 3 follows easily. Let us define a maximal time T_N^* as

$$T_N^* = \sup\{T \ge 0, \forall t \in [0,T], \|\mathcal{F}(t,\cdot)\|_\infty + \|\partial_x\mathcal{F}(t,\cdot)\|_\infty \le \varepsilon\}.$$

We notice that thanks to (18) we already have that T_N^* is positive. Indeed, (18) implies $\eta(0) \le \epsilon$. Let us define T_N and ε such that

$$\varepsilon = \frac{4CM_2 M_0^{\frac{1}{3}} T_N^{\frac{1}{3}}}{N^{\frac{1}{6}-\frac{\alpha}{3}}}, \qquad \varepsilon T_N^2 = \frac{1}{2}, \tag{43}$$

where C is exactly the number which appears in the right hand side of (20). We have focused on the right hand side of (20) since it gives the worse term for large times. The second condition is designed to match (17). This yields the expressions

$$\varepsilon = \frac{(4CM_2)^{\frac{6}{7}} M_0^{\frac{2}{7}}}{2^{\frac{1}{7}} N^{\frac{1}{7}-\frac{2\alpha}{7}}}, \qquad T_N = \frac{N^{\frac{1}{14}-\frac{\alpha}{7}}}{2^{\frac{6}{14}}(4CM_2)^{\frac{3}{7}} M_0^{\frac{1}{7}}}. \tag{44}$$

Note that T_N is large and ε small when N is large. Now let us assume that $T_N > T_N^*$. Then we have

$$\varepsilon (T_N^*)^2 + \varepsilon T_N^* \le \varepsilon T_N^2 + \varepsilon T_N \le \frac{1}{2} + \frac{\sqrt{\varepsilon}}{2} < 1 \tag{45}$$

for N sufficiently large. Consequently, we can use Lemma 4 and hence (20) at the time $T = T_N^*$. This yields by definition of ε and T_N

$$\begin{aligned} \eta(T_N^*) &\le \frac{\varepsilon}{4}\left(\frac{T_N^*}{T_N}\right)^{\frac{1}{3}} + \frac{C\varepsilon^3}{4(CM_2M_0^{\frac{1}{3}})^3}\frac{T_N^*}{T_N} + CM_2\left(\frac{M_0^{\frac{1}{3}}}{N^{\frac{1}{6}-\frac{\alpha}{3}}} + \frac{1}{N^{\frac{1}{2}-\alpha}}\right) \\ &< \frac{\varepsilon}{2} + CM_2\left(\frac{M_0^{\frac{1}{3}}}{N^{\frac{1}{6}-\frac{\alpha}{3}}} + \frac{1}{N^{\frac{1}{2}-\alpha}}\right) \\ &< \varepsilon \end{aligned} \tag{46}$$

for N sufficiently large. But this contradicts the definition of T_N^*. Indeed, by continuity, the estimate $\eta(t) \le \varepsilon$ will persist for times larger than T_N^*. Consequently, we must have $T_N \le T_N^*$. This also implies that (42) is true for $T = T_N$. This ends the proof.

3 Extensions

In this section, we state a result analogous to Theorem 3 which is valid in higher dimension, and when $d = 1$ and f_0 is flat near in a neighborhood of $|v| = 0$. The first step is to state the analogous of Lemma 2 in higher dimensions:

Lemma 7 *For every $\alpha \in (0, 1/2d)$ there exists a set $E \subset (\mathbb{T}^d \times \mathbb{R}^d)^{\mathbb{N}}$ with $P((\mathbb{T}^d \times \mathbb{R}^d)^{\mathbb{N}} \setminus E) = 0$ and a constant $C > 0$ such that for every sequence $(X_i(0) = (x_i(0), v_i(0))_{i\in\mathbb{N}}) \in E$, we have for every bounded Lipschitz function ψ*

$$\left| \frac{1}{N}\sum_{i=1}^N \psi(X_i(0)) - \int_{x,v} \psi(x,v) f_0(v)\,dx\,dv \right| \le \frac{C\|\psi\|_{BL^*}}{N^{\frac{1}{2d}-\alpha}}. \tag{47}$$

Again we refer for example to [8] for the proof. Next we can follow for large times the dynamics of systems of particles which are initially in E:

Theorem 8 *Under the same assumptions on f_0 and V as in Theorem 3, for $d=2,3$, and for $d=1$ under the additional assumption that the Lebesgue measure of the set where $f_0(v)=\|f_0\|_\infty$ is positive, there exists $C_{d,1}>0$, $C_{d,2}>0$ such that for every $(X_i^0)_i \in E$ we have the estimate*

$$\|\mu_N(t) - f_0 dxdv\|_{BL^*} \le C_{d,1}\left(\frac{1+M_0^{\frac{1}{2}}t^{\frac{1}{2}}}{N^{\frac{1}{4d}-\frac{\alpha}{2}}} + \frac{t}{N^{\frac{1}{2d}-\alpha}}\right), \quad \forall t \in [0, T_N^*],$$

with

$$T_N^* \ge C_{d,2}\frac{N^{\frac{1}{10d}-\frac{\alpha}{5}}}{M_2^{\frac{4}{5}}M_0^{\frac{1}{5}}}.$$

Note that for $d=1$, this theorem applies in particular to the stationary distribution $f_0=\mathbf{1}_{|v|\le 1}$. In this case, we find that T_N^* is of order $N^{\frac{1}{10}}$.

We shall not detail the proof of this theorem since it follows the same lines as the proof of Theorem 3. The crucial estimates used in the proof are (9), (28) of Lemma 5, and (32) of Lemma 6 which leads to (33). We have already stated in Lemma 7 how (9) has to be modified, (28) remains valid, so finally we only need to explain how (33) has to be changed. Thanks to Lemma 3 of [21], the estimate (33) has to be changed in

$$\|f(t)-f_0\|^2_{L^1(x,v)} \le C(E_K(f)(t) - E_K(f_0)), \quad \forall t. \tag{48}$$

Note that the power of the L^1 norm is a square and not a cube. Actually in Lemma 3 of [21] the case $d=1$ and f_0 flat in the vicinity of 0 is not stated even if their proof allows to get the result with a minor modification. We sketch a proof of this case in the Appendix.

The same considerations as in the end of the proof of Theorem 3, see (43–46), leads to the definitions

$$\varepsilon = \frac{T_N^{\frac{1}{2}}}{N^{\frac{1}{4d}-\frac{\alpha}{2}}}, \qquad \varepsilon T_N^2 = 1/2$$

so that we get the expression above for T_N^*.

4 Vortex Blobs in Dimension 2

Another classical problem of mean-field limit is the convergence of the point vortex system towards the incompressible Euler equations in vorticity form, see [21] and references therein. From a physical point of view, this model is important since it plays a role in the understanding of the two-dimensional turbulence. Moreover it is also used to design numerical methods and in this case the singular kernel is often smoothed, see [14, 21] and reference therein. This method is called the vortex blob method because the process of smoothing the kernel is equivalent to replace point vortices by blobs of finite size. In this paper, we will consider the regularized model: we consider a regularized version of the two-dimensional Euler equations in vorticity form

$$\partial_t\omega + (K\star\omega)\cdot\nabla\omega = 0, \quad t>0, \qquad x\in\mathbb{R}^2, \tag{49}$$

with

$$K = \nabla^\perp(V(|x|)), \quad V(|x|) = \ln|x|\chi(|x|), \tag{50}$$

where χ is a smooth nonnegative function such that $\chi(x) \in [0, 1]$ and

$$\chi(|x|) = 1 \quad \text{if } |x| \geq 2\epsilon,$$
$$\chi(|x|) = 0 \quad \text{if } |x| < \epsilon.$$

Note that in this section, since we work on the whole $\mathbb{R}^2$, we do not require the potential V to be periodic any more.

By analogy with the standard Euler equations, we shall set $u = K \star \omega$. Note that u is divergence free. The particles system associated to (49) is

$$\dot{x}_i(t) = \frac{1}{N} \sum_{j \neq i} K(x_i(t) - x_j(t)), \quad i = 1, \ldots, N. \tag{51}$$

Again, the empirical measure

$$\mu_N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i(t)}$$

is a measure solution of (49) and we are interested in the time scale for which this measure approximates a stable stationary solution of (49). In a recent paper [9], it has been proposed an equation to describe the $o(N)$ correction to the 2D Euler equations for the vortex model. This term would give a non trivial evolution on the set of the stationary solutions of the Euler equations, and is proportional to $1/N$. Here we shall prove rigorously that the stable stationary solutions of the Euler equation are metastable states for the vortex system. This is in agreement with [9]. More precisely, as in the case of the Vlasov equation, we focus on the following problem. If initially the particles are independently extracted from a stable stationary distribution ω_0 of (49), for how long time the solution of (51) will be close to ω_0?

We consider here the case where $\omega_0 \geq 0$ is compactly supported. Note that as for the case studied in Sect. 2, the phase space is 2-dimensional. We assume that $\omega_0 dx$ is a probability measure on $\mathbb{R}^2$ and hence we can define P and E as in Sect. 2 so that we get for every initial choice $(x_i^0)_{i \in \mathbb{N}} \in E$ that

$$\|\mu_N(0) - \omega_0 dx\|_{BL^*} \leq \frac{C}{N^{\frac{1}{2-\alpha}}}. \tag{52}$$

Moreover, since we consider the case where ω_0 is compactly supported, we can also assume that

$$E \subset (\operatorname{Supp} \omega_0)^{\mathbb{N}} \subset \{|x| \leq S\}^{\mathbb{N}}, \tag{53}$$

for some $S > 0$.

We can prove the following result:

Theorem 9 *Assuming that $\omega_0 = g(|x|) \geq 0 \in L^\infty$ is compactly supported with g nonincreasing and that $K \in \mathcal{C}^{1,1}(\mathbb{R})$, there exists $C_1 > 0$, $C_2 > 0$ such that for every $(x_k^0)_{k \in \mathbb{N}} \in E$, we have the estimate*

$$\|\mu_N(t) - \omega_0 dx\|_{BL^*} \leq C_1 \left(\frac{1}{N^{\frac{1}{4} - \frac{\alpha}{2}}} + \frac{t}{N^{\frac{1}{2} - \alpha}} + \frac{t^{\frac{1}{2}}}{N^{\frac{1}{4} - \frac{\alpha}{2}}} \right), \quad \forall t \in [0, T_N^*]$$

with

$$T_N^* \geq C_2 \frac{N^{\frac{1}{18}-\frac{\alpha}{9}}}{M_2^{\frac{1}{9}} M_1^{\frac{1}{9}}},$$

where $M_2 = \|K\|_{\mathcal{C}^{1,1}}$, $M_1 = \|K\|_{\mathcal{C}^1}$.

4.1 Proof of Theorem 9

The method of the proof is quite similar to the proof of Theorem 2, so that we shall only sketch it, giving the main differences.

Let us define the velocity field created by the particles

$$U(t,x) = \frac{1}{N}\sum_{i=1}^{N} K(x - x_i(t)),$$

we shall consider $\omega(t,x)$ the solution of the transport equation

$$\partial_t \omega + U \cdot \nabla \omega = 0 \tag{54}$$

with initial datum $\omega(0,x) = \omega_0(x)$. The part played by $f(t,x,v)$ in the proof of Theorem 3 is played here by ω. We shall denote respectively the moment of inertia for the continuous density $\omega(t)$ and for a particles system as

$$M(\omega(t)) = \int_{\mathbb{R}^2} \omega(t,x)|x|^2 dx, \qquad M(X^N(t)) = \frac{1}{N}\sum_{i=1}^{N} x_i^2.$$

The part played by the kinetic energy in the proof of Theorem 3 is played here by the moment of inertia. In particular the moment of inertia is preserved by both the evolutions with the particle system (51) and by the regularized Euler equations (49, 50). The bootstrap argument is on $\eta(t)$ defined by

$$\eta(t) := \|U(t,\cdot) - u_0\|_\infty + \|\nabla U(t,\cdot) - \nabla u_0\|_\infty.$$

Again, the crucial step is to prove the analogous of Lemma 4. Let $T > 0$ and $\varepsilon > 0$ such that

$$C M_1^2 M_2 T^2 (1 + T^2)\varepsilon \leq \frac{1}{2}. \tag{55}$$

If we have the estimate

$$\eta(t) \leq \varepsilon, \quad \forall t \in [0,T] \tag{56}$$

then the estimate

$$\eta(t) \leq C M_2 \left(\frac{1}{N^{\frac{1}{4}-\frac{\alpha}{2}}} + \frac{t}{N^{\frac{1}{2}-\alpha}} + \frac{t^{\frac{1}{2}}}{N^{\frac{1}{4}-\frac{\alpha}{2}}} \right) \tag{57}$$

holds for $t \in [0,T]$.

We shall not give the complete proof of this crucial step, but we shall explain the two main ingredients. As before, we can write

$$\begin{aligned}\|\mu_N(t) - \omega_0 dx\|_{BL^*} &\leq \|\mu_N(t) - \omega(t)dx\|_{BL^*} + \|\omega(t) - \omega_0\|_{L^1} \\ &= I + II.\end{aligned}$$

Now let us consider the flow $\Phi_{t,s}$ of the differential equation

$$\dot{x}(t) = U(t, x) \tag{58}$$

this means that $t \to \Phi_{t,s}(x)$ is the solution of (58) such that $\Phi_{s,s}(x) = x$. Then we have $x_i(t) = \Phi_{t,0}(x_i^0)$ and $\omega(t, x) = \omega_0(\Phi_{t,0}^{-1}(x))$, moreover since U is divergence free, $\Phi_{t,s}$ is measure preserving. Consequently, to estimate I as in the proof of Theorem 3, we just need the analogous of Lemma 5. Here, we have

Lemma 10 *Under the assumption* (56, 55), *we have the estimates*

$$\|\Phi_{t,s}\|_\infty \le S + 1, \quad \forall s, t, s \le t \le T \tag{59}$$

and

$$\|\Phi_{t,s}\|_{BL} \le CM_1(1 + t - s), \quad \forall s, t, s \le t \le T. \tag{60}$$

The proof of the lemma is postponed to the Appendix. With this lemma, we easily get

$$I \le C\frac{1+t}{N^{\frac{1}{2}-\alpha}}, \quad \forall t \in [0, T]. \tag{61}$$

To estimate II, we can use Lemma 1 in [19] to get

$$C\|\omega(t) - \omega_0\|_{L^1}^2 \le M(\omega(t)) - M(\omega_0) \tag{62}$$

and then we write

$$\begin{aligned} CII^2 &\le M(\omega(t)) - M(\omega_0) \\ &\le M(\omega(t)) - M(X^N(t)) + M(X^N(t)) - M(\omega_0) \\ &\le M(\omega(t)) - M(X^N(t)) + M(X^N(0)) - M(\omega_0) \\ &:= D_1 + D_2, \end{aligned}$$

where we have used the conservation of the moment of inertial for the particles systems. It is now very easy to estimate D_1 and D_2 thanks to arguments already used, we shall not give more details. This yields an estimate of II. By combining this estimate with (61), we easily get (57).

Once (57) is proved, we can easily end the proof of Theorem 9. We can define T_N^* as the maximum time for which the estimate (56) holds. A continuation argument leads to the choices

$$\varepsilon = \frac{CM_2T_N^{\frac{1}{2}}}{N^{\frac{1}{4}-\frac{\alpha}{2}}}, \qquad CM_2M_1^2\varepsilon T_N^4 = 1$$

which finally gives a bound from below for T_N^*.

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Appendix

Here we give the proof of Lemma 5, of Lemma 10, and, for the sake of completeness, we sketch a proof of Lemma 6 which is taken from Marchioro and Pulvirenti [20].

5.1 Proof of Lemma 5

The proof is very elementary. At first, the integration of (24) gives

$$|v(t, X_0)| \le |v_0| + \int_0^t \mathcal{F}(s, x(s))ds \le S + \varepsilon T \le S + 1.$$

For the second inequality, we have used (19) and for the last one we have used (17). Next, thanks to Duhamel formula, we can rewrite (24) as

$$\Phi_t(X_0) = e^{tA} X_0 + \int_0^t e^{(t-s)A} G(s, \Phi_s(X_0))ds,$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad G(s, \Phi_s(X_0)) = (0, \mathcal{F}(s, \Phi_s(X_0))^t.$$

Consequently, we get for $t \in [0, T]$

$$\begin{aligned} \|\Phi_t\|_{\text{Lip}} &\le (1+t) + \int_0^t (1+t-s)\|\mathcal{F}(s,\cdot)\|_{\text{Lip}}\|\Phi_s\|_{\text{Lip}} \\ &\le (1+t) + \varepsilon(1+t)\int_0^t \|\Phi_s\|_{\text{Lip}} ds, \end{aligned}$$

where we have again used (19). Finally, we get thanks to a variation on the Gronwall Lemma that

$$\|\Phi_t\|_{\text{Lip}} \le (1+t)e^{t(1+t)\varepsilon} \le C(1+t), \quad \forall t \in [0, T],$$

where in the last inequality we have used (17).

5.2 Proof of Lemma 6

We shall give the proof of the estimate (48) in the case where $d = 1$ and $\mu_0 = \text{Leb}\{f_0 = \|f_0\|_\infty\} > 0$ which is clear from the proof in [20] even if it is not stated. Following [20], we first define

$$A^k = \left\{ f > \frac{k}{M}\|f_0\|_\infty \right\}, \quad A_0^k = \left\{ f_0 > \frac{k}{M}\|f_0\|_\infty \right\}, \quad k = 1, \ldots, M-1.$$

Note that by assumption, we have $\text{Leb}(A^k) = \text{Leb}(A_0^k)$. We approximate f and f_0 by

$$f^M = \frac{\|f\|_\infty}{M}\sum_{k=1}^M \mathbf{1}_{A^k}, \qquad f_0^M = \frac{\|f\|_\infty}{M}\sum_{k=1}^M \mathbf{1}_{A_0^k}.$$

We have

$$E_K(f^M) - E_K(f_0^M) = \frac{\|f_0\|_\infty}{2M} \sum_k \int_{A^k \setminus A_0^k} v^2 - \int_{A_k^0 \setminus A^k} v^2$$
$$\geq \frac{\|f_0\|_\infty}{2M} \int_{\mu_k}^{\mu_k+\beta_k} v^2 - \int_{\mu_k-\beta_k}^{\mu_k} v^2,$$

where $\mu_k = \frac{1}{2}\,\mathrm{Leb}(A^k)$ and $\beta_k = \|\mathbf{1}_{A^k} - \mathbf{1}_{A_0^k}\|_{L^1}$. Consequently, we get

$$E_K(f^M) - E_K(f_0^M) \geq \frac{\|f_0\|_\infty}{2M} \sum_k 2\mu_k \beta_k^2$$
$$\geq \frac{\|f_0\|_\infty}{4M} \mu_0 \sum_k \|\mathbf{1}_{A^k} - \mathbf{1}_{A_0^k}\|_{L^1}^2 \geq \frac{\mu_0}{4\|f_0\|_\infty} \|f^M - f_0^M\|_{L^1}^2.$$

We finish the proof by taking the limit $M \to +\infty$.

5.3 Proof of Lemma 10

It remains to prove Lemma 10. This is more difficult than Lemma 5 in the proof of Theorem 3 since the ordinary differential equation (58) is not a small perturbation of a linear one. A proof has been given in [8]. We give here a slightly different one.

The first step is to get some useful bounds on u_0. At first, we notice that since K is smooth and ω_0 compactly supported, we have

$$\|u_0\|_\infty \leq CM_0, \qquad \|\nabla u_0\|_\infty \leq CM_1, \qquad \|\nabla u_0\|_{\mathrm{Lip}} \leq CM_2. \tag{63}$$

Moreover, by using the shape of V given by (50), we notice that

$$u(0) = -\int_{\mathbb{R}^2} V'(|y|) \frac{y^\perp}{|y|} \omega_0(|y|)dy = 0$$

by symmetry. Consequently, thanks to (63), we also have

$$|u_0(x)| \leq CM_1|x|, \quad \forall x \tag{64}$$

which gives a better estimate than (63) close to zero. Finally, we notice that we can write u_0 as $u_0 = \nabla^\perp \psi_0$ where

$$\psi_0 = \int_{\mathbb{R}^2} V(|x-y|)\omega_0(|y|)dy.$$

Since V and ω_0 are invariant by rotations, we find that ψ_0 is actually a function of $|x|$ only so that u_0 can be written under the form

$$u_0(x) = f(|x|)\frac{x^\perp}{x}. \tag{65}$$

To have lighter notations during the proof we shall denote $\Phi_{t,s}(x)$ (see (58)) by $X(t)$.

We begin with the proof of (59). Note that

$$\dot{X}(t) = u_0(X(t)) + (U(t, X(t)) - u_0(X(t)))$$

and that thanks to (65), we have $u_0(X) \cdot X = 0$. Consequently, we find thanks to (56)

$$\frac{d}{dt}\frac{1}{2}|X(t)|^2 \leq \varepsilon |X(t)|, \quad \forall t \in [s, T]$$

and hence the integration gives (59).

We now turn to the proof of the Lipschitz estimate. We shall actually prove a C^1 estimate giving a bound of $D_x \Phi_{t,s}$. Let us set for every $h \in \mathbb{R}^2$, $Y(t) = D_x \Phi_{t,s}(x) \cdot h$, Y will be the solution of the differential equation

$$\dot{Y}(t) = D_x U(t, X(t)) \cdot Y(t)$$

with the initial condition $Y(s) = h$. Let us introduce $X^0(t)$, the solution of the ordinary differential equation

$$\dot{X} = u_0(X) \tag{66}$$

with initial value $X^0(s) = x$ the same as for X. To estimate Y we notice that

$$\dot{Y} = D_x u_0(X^0(t)) \cdot Y + (J_1(t) + J_2(t)) \cdot Y \tag{67}$$

where

$$J_1(t) = D_x u_0(X(t)) - D_x u_0(X^0(t)),$$
$$J_2(t) = D_x U(t, X(t)) - D_x u_0(X(t)).$$

We want to rewrite (67) by using the Duhamel formula. The first step is to study the flow $\Phi_t^0(x)$ of the autonomous differential equation (66). Again $t \to \Phi_t^0(x)$ is the solution of (66) such that $\Phi_0^0(x) = x$. This flow has a very simple expression thanks to (65). Indeed, by identifying $\mathbb{R}^2$ and $\mathbb{C}$, we get that

$$\Phi_t^0(x) = x \exp\left(it \frac{f(|x|)}{|x|}\right).$$

This implies that for every x, $|x| \leq S$, we have

$$\begin{aligned} |D\Phi_t^0(x) \cdot h| &\leq C|h| \left(1 + t\left(f'(|x|) + \frac{f(|x|)}{|x|}\right)\right) \\ &\leq CM_1(1+t)|h| \end{aligned} \tag{68}$$

where in the last inequality, we have used (65), (63) and (64). Since the fundamental matrix $G(t,s)$, with $G(s,s) = Id$, of the linear equation

$$Z' = D_x u_0(X^0(t)) \cdot Z$$

is given by $D_x \Phi_{t-s}^0(X^0(t))$, we can rewrite (67) as

$$Y(t) = D_x \Phi_{t-s}^0(X^0(t)) \cdot h + \int_s^t D_x \Phi_{t-\tau}^0(X^0(t)) \cdot (J_1(\tau) + J_2(\tau)) d\tau. \tag{69}$$

We need to estimate J_1 and J_2. The estimate of J_2 is actually easy to get. Indeed, thanks to (56) we have

$$|J_2(\tau)| \leq \varepsilon, \quad \forall \tau \in [0, T]. \tag{70}$$

To estimate J_1, we need an estimate of $X - X^0$. Towards this, we notice that $X - X^0$ is a solution of the differential equation

$$\frac{d}{dt}(X - X^0) = D_x u_0(X^0(t)) + K_1 + K_2,$$

where

$$K_1 = U(t, X(t)) - u_0(X(t)),$$

$$K_2 = u_0(X(t)) - u_0(X^0(t)) - D_x u_0(X^0(t)) \cdot (X(t) - X^0(t)).$$

Again thanks to the Duhamel formula (we recall that X and X^0 have the same initial values), we can write

$$X(t) - X^0(t) = \int_s^t D_x \Phi^0_{t-\tau}(X^0(t)) \cdot (K_1(\tau) + K_2(\tau)) d\tau. \tag{71}$$

By (56), we have

$$|K_1(\tau)| \le \varepsilon, \quad \forall \tau \in [0, T],$$

and by the Taylor formula, we have

$$|K_2(\tau)| \le C M_2 |X(\tau) - X^0(\tau)|^2,$$

so that by using the estimate (68), we get

$$\begin{aligned} |X(t) - X^0(t)| &\le C M_1 \varepsilon (t - s)(1 + t - s) \\ &\quad + C M_2 M_1 \int_s^t (1 + |t - \tau|) |X(\tau) - X^0(\tau)|^2 d\tau. \end{aligned} \tag{72}$$

The integration of this inequality gives (we postpone the proof until the end of the section)

$$|X(t) - X^0(t)| \le C M_1 \varepsilon (t - s)(1 + (t - s)), \quad \forall t, s, 0 \le s \le t \le T \tag{73}$$

if

$$\varepsilon C M_2 M_1^2 T^2 (1 + T)^2 \le \frac{1}{2}. \tag{74}$$

Thanks to (73), we get

$$|J_1| \le C M_2 M_1 \varepsilon (t - s)(1 + (t - s)), \quad \forall t, s, 0 \le s \le t. \tag{75}$$

By choosing T in such a way that the constraint (74) is verified, it is now possible to estimate Y. Using (69) with the estimates (68), (75), (70), and (73), we get

$$Y(t) \le C M_1 (1 + t - s)|h| + C \varepsilon M_1 M_2 (1 + (t - s)^2) \int_s^t Y(\tau) d\tau.$$

A variation on the Gronwall Lemma then gives

$$|Y(t)| \le C M_1 M_2 (1 + t)|h| e^{C \varepsilon M_1^2 t (1 + t^2)}.$$

By using (55), we finally get the desired result.

It remains to explain how we get (73) from (72). Note that (72) implies that we have for every $t_1 \geq t$ and $t_1 \leq T$ that

$$\begin{aligned} |X(t) - X^0(t)| &\leq CM_1\varepsilon(t_1 - s)(1 + t_1 - s) \\ &\quad + CM_2M_1(1 + t_1 - s)\int_s^t |X(\tau) - X^0(\tau)|^2 d\tau. \end{aligned} \tag{76}$$

Now let us define

$$Z(t) = CM_1\varepsilon(t_1 - s)(1 + t_1 - s) + CM_2M_1(1 + t_1 - s)\int_s^t |X(\tau) - X^0(\tau)|^2.$$

Note that

$$Z(s) = CM_1\varepsilon(t_1 - s)(1 + t_1 - s). \tag{77}$$

Next we get

$$\begin{aligned} \dot{Z}(t) &\leq CM_2M_1(1 + t_1 - s)|X(t) - X^0(t)|^2 \\ &\leq CM_2M_1(1 + t_1 - s)|Z(t)^2. \end{aligned}$$

The integration of this Riccati inequality gives

$$Z(t) \leq \frac{Z(s)}{1 - CM_2M_1(t - s)(1 + t - s)Z(s)}, \quad \forall t, s \leq t \leq t_1 \leq T.$$

Thanks to (77), we get in particular

$$\begin{aligned} |X(t_1) - X^0(t_1)| &\leq Z(t_1) \\ &\leq \frac{CM_1\varepsilon(t_1 - s)(1 + t_1 - s)}{1 - CM_2M_1^2(t_1 - s)^2(1 + t_1 - s)^2}. \end{aligned}$$

Since this inequality is actually valid for every t_1, we get (73) under the condition (55).

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